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An Implicit Scheme for Nonlinear Evolution Equations

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1. INTRODUCTION

It is well known that nonlinear instability may occur when partial differential equations such as the advection equation, Burgers' equation, the KdV equation, and the generalized KdV equation are approximated by finite difference schemes, even if the corresponding linearized equations are stable. Philips [1], Arakawa [2], Richtmyer [3], Fornberg [11], and Majda [12] studied nonlinear instability and showed that linearized analysis and constant-coefficient analysis may fail to predict instability.

In this paper, we show that for a class of evolution equations there exists an implicit difference scheme which is nonlinearly stable without any conditions on $\Delta t, \Delta x$.

From the following example concerned with numerical integration of ordinary differential equations, we can get some useful information: Consider an ordinary differential equation

$$\dot{y} = F(t, y). \tag{1.1}$$

If we can rewrite it in the form

$$\dot{y} = f(t, y)\, y \tag{1.2}$$

and if $f(t, y) \leqslant 0$ for all $0 \leqslant t \leqslant T$, $a \leqslant y \leqslant b$, and $y^* = y^n$, then the implicit scheme

$$(y^{n+1} - y^n)/\Delta t = f(t, y^*)(\alpha y^{n+1} + (1-\alpha)\, y^n) \tag{1.3}$$

is absolutely stable when $\alpha \geqslant \frac{1}{2}$.

From (1.3) we have

$$y^{n+1}(1 - \Delta t\, \alpha f(t, y^n)) = (1 + \Delta t(1-\alpha) f(t, y^n))\, y^n$$

or

$$y^{n+1} = (1 + (1-\alpha)\,\Delta t f(t, y^n))/(1 - \alpha\,\Delta t f(t, y^n))\, y^n = S y^n. \tag{1.4}$$

Note that $|S| \leqslant 1$ when $\alpha \geqslant \frac{1}{2}$.

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This scheme is of first-order accuracy, even if $\alpha = \frac{1}{2}$. If $y^* = (y^{n+1} + y^n)/2$, it is of second-order accuracy, but is nonlinear. Therefore it is not convenient to apply it.

We now apply the predictor–corrector procedure

$$y^* = y^n + \Delta t\,\beta f(t, y^n)\,y^n, \qquad y^{n+1} = y^n + \Delta t\,f(t, y^*)(\alpha y^{n+1} + (1-\alpha)\,y^n)/2. \quad (1.5)$$

It is evident that this scheme has second-order accuracy when $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$. Moreover, it is absolutely stable because $f(t, y^*)$ is nonpositive. Then (1.5) is linear in y^{n+1} and hence is very easy to solve.

We can extend the result obtained in the above example to a general operator equation.

2. Notation and Preliminaries

In this section, we give some definitions and preliminary lemmas. Let Δt and h be the increments of the time and space variables, respectively. Let u_m^n denote the value of the mesh function $u(n\,\Delta t, mh)$ at the point $x = mh$ and $t = n\,\Delta t$, where n and m are positive integers.

We define the inner product

$$(u, v) = h \sum_{m=1}^{N} u_m^n \cdot v_m^n \quad (2.1)$$

and norm

$$\|u\|^2 = (u, u). \quad (2.2)$$

The forward, centered, and backward difference approximations are denoted respectively by Δ_+, Δ_0, Δ_- of $\partial/\partial x$.

$$u_x = \Delta_+ u_m^n = (u_{m+1}^n - u_m^n)/h, \quad (2.3)$$

$$u_{\dot{x}} = \Delta_0 u_m^n = (u_{m+1}^n - u_{m-1}^n)/2h, \quad (2.4)$$

$$u_{\bar{x}} = \Delta_- u_m^n = (u_m^n - u_{m-1}^n)/h, \quad (2.5)$$

$$u_t = \Delta_+^t u_m^n = (u_m^{n+1} - u_m^n)/\Delta t. \quad (2.6)$$

If u_m^n and v_m^n are periodic in m with period N, we have

$$(\Delta_+ u, v) = -(u, \Delta_- v), \quad (2.7)$$

$$(\Delta_- u, v) = -(u, \Delta_+ v), \quad (2.8)$$

$$(\Delta_0 u, v) = -(u, \Delta_0 v). \quad (2.9)$$

The proof of Lemmas 1–7 is simple and is therefore omitted.

LEMMA 1. $2(u, u_t) = (\|u\|^2)_t - \Delta t \|u_t\|^2$.

LEMMA 2. $(u, u_{x\bar{x}}) = -\|u_{\bar{x}}\|^2$.

LEMMA 3. $(u, \Delta_+^r \Delta_-^r u) = (-1)^r \|\Delta_-^r u\|^2$.

LEMMA 4. $(u, A_{(q+1)/(q+2)}(v^*)u) + (v, A_{(q+1)/(q+2)}(u^*)u) = 0$,

where $A_{(q+1)/(q+2)} = (1/(q+2))\, u^q \Delta_0 \cdot + (1/(q+2))\, \Delta u^q \cdot$.

LEMMA 5. $(u, \Delta_+ \Delta_0 \Delta_- u) = 0$.

LEMMA 6. (1) $2(u, \Delta_+^r \Delta_-^{r+1} u) = (-1)^r h \|\Delta_-^{r+1} u\|^2$.
(2) $2(u, \Delta_+^{r+1} \Delta_-^r u) = (-1)^{r+1} h \|\Delta_-^{r+1} u\|^2$.

LEMMA 7. $(u, \Delta_+^r \Delta_0 \Delta_-^r u) = 0$.

3. THEOREMS

We consider the nonlinear evolution equation $\dot{u} = F(t, u)$. Let it be rewritten in the form

$$\partial u(t)/\partial t = A_\theta(t, u)u, \qquad 0 < t \leqslant T, \tag{3.1a}$$

$$u(0) = u_0, \tag{3.1b}$$

where A_θ is in general a nonlinear differential operator and θ is a real parameter which will be used below.

We shall discuss initial value problem (3.1) in L_2. The boundary condition, if any, will be periodic.

Suppose that for every parameter θ we have

$$(A_\theta(t, u)u, u) \leqslant 0, \tag{3.2}$$

so that the operator A_θ is nonpositive. Multiplying (3.1a) by u, intergrating over space x, and making use of (3.2), we get

$$(\partial/\partial t)(u(t), u(t)) \leqslant 0. \tag{3.3}$$

This means that the solution of Eq. (3.1) is continuously dependent on the initial value.

In order for the difference scheme of Eq. (3.1) to be stable for a long time integration, we must choose a proper parameter θ so that the corresponding difference operator $\mathscr{A}_\theta$ satisfies a condition analogous to (3.2), namely,

$$(\mathscr{A}_\theta(t, u^*)u, u) \leqslant 0. \tag{3.3$'$}$$

Applying the backward Euler integration formula to Eq. (2.1), we obtain

$$u_m^{n+1}(t) = u_m^n(t) + \Delta t\, \mathscr{A}_\theta(t, u_m^{n+1})\, u_m^{n+1} \tag{3.4}$$

If the trapezoid integration formula of Eq. (2.1) is used, we obtain

$$u_m^{n+1}(t) = u_m^n(t) + \Delta t\, \mathscr{A}_\theta(t, (u_m^{n+1} + u_m^n)/2)(u_m^{n+1} + u_m^n)/2. \tag{3.5}$$

Difference equations (3.4) and (3.5) are nonlinear, however, and not easy to solve. Linearizing Eqs. (3.4) and (3.5), we get

$$u_m^{n+1}(t) = u_m^n(t) + \Delta t\, \mathscr{A}_\theta(t, u^*)\, u_m^{n+1}, \tag{3.6}$$

$$u_m^{n+1}(t) = u_m^n(t) + \Delta t\, \mathscr{A}_\theta(t, u^*)(u_m^{n+1} + u_m^n)/2, \tag{3.7}$$

where u^* is known.

THEOREM A. *Difference equations* (3.6), (3.7) *are absolutely stable if the difference operator* $\mathscr{A}_\theta$, *which is the approximation of differential operator* A_θ, *satisfies condition* (3.3′) *for all* t, $0 \leqslant t \leqslant T$, $u^n \in C$.

Proof. Multiplying (3.6) by u_m^{n+1} and summing over all m, we get

$$(u^{n+1}, u^{n+1}) = (u^n, u^{n+1}) + (\mathscr{A}_\theta(t, u^*)\, u^{n+1}, u^{n+1}).$$

Noting (3.3′), we then have an energy inequality

$$\|u^{n+1}\| \leqslant \|u^n\| \tag{3.8}$$

by Schwarz's inequality.

For the scheme (3.7), we multiply (3.7) by $(u_m^{n+1} + u_m^n)$ and sum over all m to get

$$((u^{n+1} - u^n), (u^{n+1} + u^n))^2 = (\mathscr{A}_\theta(t, u^*)(u^{n+1} + u^n)/2, (u^{n+1} + u^n)/2).$$

Noting (3.3′), we get the same energy inequality

$$\|u^{n+1}\| \leqslant \|u^n\|, \tag{3.9}$$

and the proof of theorem is completed.

Remark. For scheme (3.7), the equality holds in (3.9) provided condition (3.3′) is satisfied with equality.

Now we take the predictor–corrector procedure

$$u_m^* = u_m^n + \Delta t\, \beta \mathscr{A}_\theta(t, u^n)\, u_m^n, \qquad u_m^{n+1} = u_m^n + \Delta t\, \mathscr{A}_\theta(t, u^*)(\alpha u_m^{n+1} + (1-\alpha)\, u_m^n). \tag{3.10}$$

For this scheme, we have

THEOREM B. *Predictor–corrector scheme* (3.10) *is absolutely stable if the difference operator* $\mathscr{A}_\theta$ *which approximates differential operator* A_θ *for some parameter* θ *satisfies condition* (3.3′) *for all* t, $0 \leqslant t \leqslant T$, $u^n \in C$. *Moreover, this scheme has second-order accuracy if* $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$.

The proof is similar to the one above. The second-order accuracy is obvious.

THEOREM C. *For the leap-frog scheme*

$$(u_m^{n+1} - u_m^{n-1})/(2\,\Delta t) = \mathscr{A}_\theta(t, u^*)\, u_m^n, \tag{3.11}$$

suppose that the difference operator $\mathscr{A}_\theta$ *satisfies condition* (3.3′) *for some* θ, *then we have a quasi-energy inequality*

$$(u^{n+1}, u^n) \leqslant \|u^0\| \qquad \textit{for all} \quad n \quad (n = 1, 2, 3, \ldots). \tag{3.12}$$

Proof. Multiplying (3.11) by u_m^n and summing over all m, noting (3.3), we obtain

$$(u^{n+1}, u^n) \leqslant (u^n, u^{n-1}).$$

We first use the forward time difference scheme

$$(u_m^1 - u_m^0)/\Delta t = \mathscr{A}_\theta(t, u^0)\, u_m^0. \tag{3.13}$$

Because of (3.3′), we have

$$(u^1, u^0) \leqslant \|u^0\|^2.$$

Therefore

$$(u^{n+1}, u^n) \leqslant (u^n, u^{n-1}) \leqslant \cdots \leqslant (u^1, u^0) \leqslant \|u^0\|^2. \tag{3.14}$$

Remark. For scheme (3.11), the quasi-energy equality holds in (3.14) provided condition (3.3′) is satisfied with equality.

4. APPLICATIONS

4.1 *The Advection Equation or Model Nonlinear Wave Equation*

Consider an advection equation

$$u_t + uu_x = 0 \tag{4.1.1}$$

which possesses an infinite number of conservation laws [4]. In order for the difference equation to be stable for long time calculations, the difference equation should be established on the basis of the energy conservation law rather than of the quality conservation law.

For Eq. (4.1.1), we take the differential operator

$$A_\theta(t, u) = (1-\theta) u \frac{\partial}{\partial x} + \frac{\theta}{2} \frac{\partial u \cdot}{\partial x}. \tag{4.1.2}$$

Obviously $(A_\theta(t, u)u, u) = 0$ for every parameter θ. The difference operator $\mathscr{A}_\theta$, which approximates the differential operator A_θ, takes the form

$$\mathscr{A}_\theta(t, u) = (1-\theta) u \Delta_0 + (\theta/2) \Delta_0 u \cdot. \tag{4.1.3}$$

By Lemma 4 in Section 2, it is not difficult to prove

$$(\mathscr{A}_{2/3}(t, u^*) u, u) = 0. \tag{4.1.4}$$

Hence $\mathscr{A}_{2/3}(t, u^*)$ is nonpositive. With $u^* = u^n$, scheme (3.6) takes the form

$$u_m^{n+1} = u_m^n + (\Delta t/3)(u_m^n \Delta_0 u_m^{n+1} + \Delta_0 u_m^n u_m^{n+1}). \tag{4.1.5}$$

Using Theorem A, we have

$$\| u^{n+1} \| \leqslant \| u^n \|,$$

i.e., scheme (4.1.5) is absolutely stable.

When $u^* = u^n$, scheme (3.7) has the form

$$u_m^{n+1} = u_m^n + (\Delta t/6)(u_m^n \Delta_0 u_m^{n+1} + \Delta_0 u_m^n u_m^{n+1}) + (\Delta t/6)(u_m^n \Delta_0 u_m^n + \Delta_0 u_m^n u_m^n). \tag{4.1.6}$$

Using the remark following Theorem A, we have

$$\| u^{n+1} \| = \| u^n \| = \| u^0 \|,$$

i.e., scheme (4.1.6) is absolutely stable.

In order to arrive at the second-order accuracy, we take $\beta = \frac{1}{2}$; then (3.10) is reduced to

$$u_m^* = u_m^n + (\Delta t/6)(u_m^n \Delta_0 u_m^n + \Delta_0 u_m^n \cdot u_m^n),$$
$$u_m^{n+1} = u_m^n + (\Delta t/6)[(u_m^* \Delta_0 u_m^{n+1} + \Delta_0 u_m^* u_m^{n+1}) + (u_m^* \Delta_0 u_m^n + \Delta_0 u_m^* u_m^n)].$$

Using the remark following Theorem B, we obtain

$$\| u^{n+1} \| = \| u^n \| = \| u^0 \|.$$

A very similar result, but for a nonlinear scheme, is proved in [3, p. 142].

4.2 *Burgers' Equation*

We take the operator A_θ of (2.1) in the form

$$A_\theta(t, u) = (1-\theta) u \frac{\partial}{\partial x} + \frac{\theta}{2} \frac{\partial u \cdot}{\partial x} + v \frac{\partial^2}{\partial x^2}. \tag{4.2.1}$$

Then the corresponding difference operator has the form

$$\mathscr{A}_\theta(t, u) = (1-\theta)\, u\Delta_0 + (\theta/2)\, \Delta_0 u \cdot + v\Delta_+ \Delta_- . \tag{4.2.2}$$

Take $\theta = \frac{2}{3}$ and notice Lemmas 2 and 4. It is not difficult to prove

$$(\mathscr{A}_{2/3}(t, u^*) u, u) \leqslant 0. \tag{4.2.3}$$

Then scheme (3.6) is of the form

$$u_m^{n+1} = u_m^n + (\Delta t/3)(u_m^* \Delta_0 u^{n+1} + \tfrac{1}{3}\Delta_0 u_m^* u_m^{n+1}) + v\, \Delta t\, \Delta_+ \Delta_- u_m^{n+1}. \tag{4.2.4}$$

Upon setting $u^* = u^{n+1}$, we have

$$u_m^{n+1} = u_m^n + (\Delta t/3)(u_m^{n+1} \Delta_0 u_m^{n+1} + \tfrac{1}{3}\Delta_0 u_m^{n+1} u_m^{n+1}) + v\, \Delta t\, \Delta_+ \Delta_- u_m^{n+1}. \tag{4.2.5}$$

Expanding this equation into the form of equations with filter [5], we obtain some relations between filter parameter k and parameter θ

$k = 0$	corresponding to	$\theta = 1$,
$k = \infty$	corresponding to	$\theta = 0$,
$k = 2$	corresponding to	$\theta = \frac{1}{2}$,
$k = -1$	corresponding to	$\theta = 2$,
$k = 1$	corresponding to	$\theta = \frac{2}{3}$.

In scheme (4.2.4), i.e., $\theta = \frac{2}{3}$ or $k = 1$, let $u_m^* = u_m^n$. Using relation (4.2.3) and Theorem A, we have

$$\|u^{n+1}\| \leqslant \|u^n\|.$$

Thus scheme (4.2.4) is absolutely stable. It is evident that if $u_m^* = u_m^{n+1}$ and $\theta = \frac{2}{3}$ (i.e., $k = 1$) as in [5], then scheme (4.2.5) is also absolutely stable, but then the equation is nonlinear and not convenient to solve.

Applying scheme (3.10), we get

$$\begin{aligned} u_m^* &= u_m^n + (\Delta t\, \beta/3)(u_m^n \Delta_0 u_m^n + \Delta_0 u_m^n u_m^n) + v\beta\, \Delta t\, \Delta_+ \Delta_- u_m^n \\ u_m^{n+1} &= u_m^n + (\Delta t/6)(u_m^* \Delta_0 u_m^{n+1} + \Delta_0 u_m^* u_m^{n+1}) + v(\Delta t/2)\, \Delta_+ \Delta_- u_m^{n+1} \\ &\quad + (\Delta t/6)(u_m^* \Delta_0 u_m^n + \Delta_0 u_m^* u_m^n) + v(\Delta t/2)\, \Delta_+ \Delta_- u_m^n. \end{aligned} \tag{4.2.6}$$

With a similar argument (i.e., making use of relation (4.2.3) and applying Theorem A), we can show that scheme (4.2.6) is absolutely stable,

$$\|u^{n+1}\| \leqslant \|u^n\|.$$

If $\beta = \frac{1}{2}$, this scheme has second-order accuracy.

4.3 KdV *Equation*

$$u_t = uu_x + u_{xxx}. \tag{4.3.1}$$

This equation possesses an infinite number of conservation laws. We establish a difference equation based on the second conservation law in preference to one based on the first conservation law.

For Eq. (4.3.1), we take the differential operator

$$A_\theta(t, u) = (1-\theta)u\frac{\partial}{\partial x} + \frac{\theta}{2}\frac{\partial u\cdot}{\partial x} + \frac{\partial^3}{\partial x^3}. \tag{4.3.2}$$

$A(t, u)$ is a nonpositive operator. The difference operator A takes the form

$$\mathscr{A}_\theta(t, u) = (1-\theta)\, u\Delta_0 + (\theta/2)\Delta_0 u\cdot + \Delta_+\Delta_0\Delta_-.$$

Taking $\theta = \frac{2}{3}$ and using Lemmas 4 and 5, we get

$$(\mathscr{A}_{2/3}(t, u^*)u, u) = 0. \tag{4.3.3}$$

Applying scheme (3.6), we get [7]

$$u_m^{n+1} = u_m^n + (\Delta t/3)(u_m^n\Delta_0 u_m^{n+1} + \Delta_0 u_m^n u_m^{n+1}) + \Delta t\, \Delta_+\Delta_0\Delta_-\, u_m^{n+1}. \tag{4.3.4}$$

Using equality (4.3.3) and Theorem A, we get

$$\|u^{n+1}\| \leqslant \|u^n\|.$$

Applying scheme (3.7), we obtain [6]

$$\begin{aligned} u_m^{n+1} = u_m^n &+ (\Delta t/6)(u_m^n\Delta_0 u_m^{n+1} + \Delta_0 u_m^n u_m^{n+1}) + (\Delta t/2)\, \Delta_+\Delta_0\Delta_-\, u_m^{n+1} \\ &+ (\Delta t/6)(u_m^n\Delta_0 u_m^n + \Delta_0 u_m^n u_m^n) + (\Delta t/2)\, \Delta_+\Delta_0\Delta_-\, u_m^n. \end{aligned} \tag{4.3.5}$$

Using equality (4.3.3) and Theorem A, we get

$$\|u^{n+1}\| = \|u^n\|.$$

Using scheme (3.10), we get a predictor–corrector procedure [8]

$$\begin{aligned} u_m^* &= u_m^n + (\beta\,\Delta t/3)(u_m^n\Delta_0 u_m^n + \Delta_0 u_m^n u_m^n) + \Delta t\, \beta\Delta_+\Delta_0\Delta_-\, u_m^n, \\ u_m^{n+1} &= u_m^n + (\Delta t/6)(u_m^*\Delta_0 u_m^{n+1} + \Delta_0 u_m^* u_m^{n+1}) + (\Delta t/2)\, \Delta_+\Delta_0\Delta_-\, u_m^{n+1} \\ &\quad + (\Delta t/6)(u_m^*\Delta_0 u_m^n + \Delta_0 u_m^* u_m^n) + (\Delta t/2)\, \Delta_+\Delta_0\Delta_-\, u_m^n. \end{aligned}$$

This scheme has second-order accuracy. Applying Theorem B and equality (4.3.3), we obtain

$$\|u^{n+1}\| = \|u^n\|.$$

4.4 *The Equation*

$$u_t = u^q u_1 + u_{2r+1}. \tag{4.4.1}$$

This class of equations possesses at least three conservation laws [9].

For Eq. (4.4.1) we take the differential operator

$$A_\theta(t, u) = (1-\theta)\, u^q \frac{\partial}{\partial x} + \frac{\theta}{q+1} \frac{\partial u^q \cdot}{\partial x} + \frac{\partial^{2r+1}}{\partial x^{2r+1}}. \tag{4.4.2}$$

$A_\theta(t, u)$ is a nonpositive operator because

$$(A_\theta(t, u) u, u) = 0.$$

The difference operator $\mathscr{A}_\theta$ takes the form

$$\mathscr{A}_\theta(t \cdot u) = (1-\theta)\, u^q \Delta_0 + (\theta/(q+1))\, \Delta_0 u^q \cdot + \Delta_+^r \Delta_0 \Delta_-^r. \tag{4.4.3}$$

Taking $\theta = q + 1/q + 2$, we get

$$\mathscr{A}_{(q+1)/(q+2)}(t, u) = (1/(q+2))\, u^q \Delta_0 + (1/(q+2))\, \Delta_0 u^q \cdot + \Delta_+^r \Delta_0 \Delta_-^r. \tag{4.4.4}$$

Applying Lemmas 4 and 7, we obtain

$$(\mathscr{A}_{(q+1)/(q+2)}(t, u^*) u, u) = 0. \tag{4.4.5}$$

Applying schemes (3.6), (3.7), and (3.10), respectively, we obtain

$$u_m^{n+1} = u_m^n + (\Delta t/(q+2))[(u_m^q)^n \Delta_0 u_m^{n+1} + \Delta_0 (u_m^q)^n u_m^{n+1}] + \Delta t\, \Delta_+^r \Delta_0 \Delta_-^r u_m^{n+1}, \tag{4.4.6}$$

$$\begin{aligned} u_m^{n+1} = u_m^n &+ (\alpha\, \Delta t/(q+2))[(u_m^q)^n \Delta_0 u_m^{n+1} + \Delta_0 (u_m^q)^n u_m^{n+1}] + \Delta t\, \alpha \Delta_+^r \Delta_0 \Delta_-^r u_m^{n+1} \\ &+ (1-\alpha)(\Delta t/(q+2))[(u_m^q)^n \Delta_0 u^n + \Delta_0 (u_m^q)^n u_m^n] + (1-\alpha)\, \Delta t\, \Delta_+^r \Delta_0 \Delta_-^r u_m^n, \end{aligned} \tag{4.4.7}$$

$$\begin{aligned} u_m^* &= u_m^n + (\beta\, \Delta t/(q+2))[(u_m^q)^n \Delta_0 u_m^n + \Delta_0 (u_m^q)^n u_m^n] + \beta\, \Delta t\, \Delta_+^r \Delta_0 \Delta_-^r u_m^{n+1}, \\ u_m^{n+1} &= u_m^n + \frac{\alpha\, \Delta t}{q+2} [(u_m^q)^* \Delta_0 u_m^{n+1} + \Delta_0 (u_m^q)^* u_m^{n+1}] + \alpha\, \Delta t\, \Delta_+^r \Delta_0 \Delta_-^r u_m^{n+1} \\ &\quad + (1-\alpha) \frac{\Delta t}{q+2} [(u_m^q)^* \Delta_0 u_m^n + \Delta_0 (u_m^q)^* u_m^n] + (1-\alpha)\, \Delta t\, \Delta_+^r \Delta_0 \Delta_-^r u_m^n. \end{aligned} \tag{4.4.8}$$

Using relation (4.4.5) and the Lemmas and theorems for scheme (4.4.6), we have

$$\|u^{n+1}\| \leqslant \|u^n\|.$$

For schemes (4.4.7) and (4.4.8), taking $\alpha = \frac{1}{2}$ we have

$$\|u^{n+1}\| = \|u^n\|.$$

Moreover, scheme (4.4.8) has second-order accuracy when $\beta = \frac{1}{2}$.

For the equation

$$u_t = u^q u_1 + a u_{2r+1}, \tag{4.4.1$'$}$$

we only considered linear stability in [10]; now we consider nonlinear stability and show the following:

THEOREM D. (1) *The symmetric Crank–Nicholson scheme for* (4.4.1′) *is absolutely stable when* $\alpha \geqslant \frac{1}{2}$, $\theta = (q+1)/(q+2)$;

(2) *The right Crank–Nicholson scheme for* (4.4.1′) *is absolutely stable when* $(-1)^r \cdot a > 0$, $\alpha \geqslant \frac{1}{2}$, $\theta = (q+1)/(q+2)$;

(3) *The left Crank–Nicholson scheme for* (4.4.1′) *is absolutely stable when* $(-1)^r \cdot a < 0$, $\alpha \geqslant \frac{1}{2}$, $\theta = (q+1)/(q+2)$.

Proof. Scheme (4.4.7) can be rewritten in the form

$$u_t = \mathscr{A}_{(q+1)/(q+2)}(t, u^*)(u^n + \alpha\, \Delta t\, u_t). \tag{4.4.7$'$}$$

Multiplying (4.4.7′) by $u^n + \alpha\, \Delta t\, u_t$, and taking the inner product, from (4.4.5) we have

$$(u_t, u^n + \alpha\, \Delta t\, u_t) = 0.$$

Applying Lemma 1, we get

$$\tfrac{1}{2}(\|u^n\|^2)_t - \tfrac{1}{2}\Delta t\, \|u_t\|^2 + \alpha\, \Delta t\, \|u_t\|^2 = 0,$$

i.e., $\frac{1}{2}\|u\|_t^2 - \Delta t(\frac{1}{2} - \alpha)\|u_t\|^2 = 0$.

So when $\alpha \geqslant \frac{1}{2}$ and $\theta = (q+1)/(q+2)$ the symmetric Crank–Nicholson scheme is absolutely stable.

DEFINITION. The Crank–Nicholson scheme is called *right* if $\partial^{2r+1}/\partial x^{2r+1} \approx \Delta_+^{r+1}\Delta_-^r$.

For the right Crank–Nicholson scheme, the difference operator has the form

$$\widetilde{\mathscr{A}_{(q+1)/(q+2)}}(t, u) = (1/(q+2))\, u^q \Delta_0 + (1/(q+2))\, \Delta_0 u^q \cdot + a\Delta_+^{r+1}\Delta_-^r.$$

(See Fig. 1a.) Applying Lemmas 4 and 6(2), we obtain

$$(\widetilde{\mathscr{A}_{(q+1)/(q+2)}}(t, u^*)u, u) \leqslant 0, \qquad \text{when} \quad (-1)^{r+1} \cdot a < 0, \quad \text{i.e.,} \quad (-1)^r \cdot a > 0.$$

Multiplying (4.4.6) by $u^n + \alpha \, \Delta t \, u_t$ and taking the inner product, from (4.4.7) we have

$$(u_t, u^n + \alpha \, \Delta t \, u_t) \leqslant 0.$$

Applying Lemma 1, we get

$$\tfrac{1}{2} \|u\|_t^2 - \Delta t(\tfrac{1}{2} - \alpha) \|u_t\|^2 \leqslant 0.$$

So when $\alpha \geqslant \frac{1}{2}$, $(-1)^r \cdot a > 0$, $\theta = (q+1)/(q+2)$, the right symmetric Crank–Nicholson scheme is absolutely stable.

DEFINITION. The Crank–Nicholson scheme is called *left* if $\partial^{2r+1}/\partial x^{2r+1} \approx \Delta_+^r \Delta_-^{r+1}$.

For example, the stencil when $r = 1$ is shown in Fig. 1b.

For the left Crank–Nicholson scheme, the difference operator has the form

$$\widetilde{\mathscr{A}_{(q+1)/(q+2)}}(t, u) = (1/(q+2))\, u^q \Delta_0 + (1/(q+2))\, \Delta_0 u^q \cdot + a\Delta_+^r \Delta_-^{r+1}.$$

Applying Lemmas 4 and 6(1), we obtain

$$(\widetilde{\mathscr{A}_{(q+1)/(q+2)}}(t, u^*)u, u) \leqslant 0 \qquad \text{when} \quad (-1)^r \cdot a < 0.$$

The proof of (3) is the same.

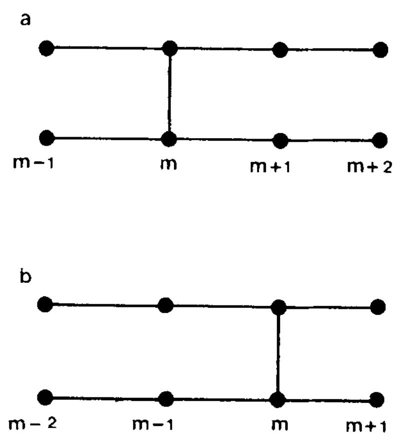

FIG. 1. (a) Stencil for the right Crank–Nicholson scheme. (b) Stencil for the left Crank–Nicholson scheme.

4.5 *The Equation*

$$u_t = u^q u_1 + (-1)^{r+1} u_{2r}. \tag{4.5.1}$$

This class of equations possesses only one conservation law [6]. For Eq. (4.5.1), we have the differential operator

$$A_\theta(t, u) = (1-\theta)\, u^q \frac{\partial}{\partial x} + \frac{\theta}{q+1} \frac{\partial u^q \cdot}{\partial x} + (-1)^{r+1} \frac{\partial^{2r}}{\partial x^{2r}}. \tag{4.5.2}$$

$A_\theta(t, u)$ is a nonpositive operator because

$$(A_\theta(t, u) u, u) \leqslant 0.$$

The difference operator $\mathscr{A}_\theta$ takes the form

$$\mathscr{A}_\theta(t, u) = (1-\theta)\, u^q \Delta_0 + (\theta/(q+1))\, \Delta_0 u^q \cdot + (-1)^{r+1} \Delta_+^r \Delta_-^r. \tag{4.5.3}$$

Taking $\theta = (q+1)/(q+2)$, we get

$$\mathscr{A}_{q+1/q+2}(t, u) = (1/(q+2))\, u^q \cdot \Delta_0 + (1/(q+2))\, \Delta_0 u^q \cdot + (-1)^{r+1} \Delta_+^r \Delta_-^r. \tag{4.5.4}$$

Applying Lemmas 3 and 4, we obtain

$$(\mathscr{A}_{(q+1)/(q+2)}(t, u^*) u, u) \leqslant 0. \tag{4.5.5}$$

Applying schemes (3.6), (3.7), and (3.10), respectively, we obtain

$$u_m^{n+1} = u_m^n + (\Delta t/(q+2))[(u_m^q)^n \Delta_0 u_m^{n+1} + \Delta_0 (u_m^q)^n u_m^{n+1}] + \Delta t\, \Delta_+^r \Delta_-^r u_m^{n+1}, \tag{4.5.6}$$

$$\begin{aligned} u_m^{n+1} = u_m^n &+ (\alpha\, \Delta t/(q+2))[(u_m^q)^n \Delta_0 u_m^{n+1} + \Delta_0 (u_m^q)^n u_m^{n+1}] \\ &+ \alpha\, \Delta t (-1)^{r+1} \Delta_+^r \Delta_-^r u_m^{n+1} \\ &+ ((1-\alpha)\, \Delta t/(q+2))[(u_m^q)^n \Delta_0 u_m^n + \Delta_0 (u_m^q)^n u_m^n] \\ &+ (1-\alpha)\, \Delta t (-1)^{r+1} \Delta_+^r \Delta_-^r u_m^n, \end{aligned} \tag{4.5.7}$$

$$\begin{aligned} u_m^* &= u_m^n + (\beta\, \Delta t/(q+2))[(u_m^q)^n \Delta_0 u_m^n + \Delta_0 (u_m^q)^n u_m^n] + \beta\, \Delta t (-1)^{r+1} \Delta_+^r \Delta_-^r u_m^n, \\ u_m^{n+1} &= u_m^n + (\alpha\, \Delta t/(q+2))[(u_m^q)^* \Delta_0 u_m^{n+1} + \Delta_0 (u_m^q)^* u_m^{n+1}] \\ &\quad + \alpha\, \Delta t (-1)^{r+1} \Delta_+^r \Delta_-^r u_m^{n+1} \\ &\quad + (1-\alpha)(\Delta t/(q+2))[(u_m^q)^* \Delta_0 u_m^n + \Delta_0 (u_m^q)^* u_m^n] \\ &\quad + (1-\alpha)\, \Delta t (-1)^{r+1} \Delta_+^r \Delta_-^r u_m^n. \end{aligned} \tag{4.5.8}$$

Scheme (4.5.7) can be rewritten in the form

$$u_t = \mathscr{A}_{(q+1)/(q+2)}(t, u^*)(u^n + \alpha\, \Delta t\, u_t). \tag{4.5.7$'$}$$

Multiplying (4.5.7′) by $u^n + \alpha \Delta t\, u_t$ and taking the inner product, from (4.5.7) we have

$$(u_t, u^n + \alpha \Delta t\, u_t) \leqslant 0.$$

Applying Lemma 1, we have

$$\tfrac{1}{2} \|u\|_t^2 - \Delta t(\tfrac{1}{2} - \alpha) \|u_t\|^2 \leqslant 0.$$

THEOREM E. *The symmetric Crank–Nicholson scheme for* (4.5.1) *is absolutely stable when* $\alpha \geqslant \frac{1}{2}$, $\theta = (q+1)/(q+2)$.

Remark. Predictor–corrector scheme (4.5.8) is absolutely stable when $\alpha \geqslant \frac{1}{2}$ and $\theta = (q+1)/(q+2)$, and has second-order accuracy wen $\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$.

5. Numerical Computations

In the present section, we discuss only Burgers' equation

$$u_t = uu_x + \nu u_{xx}. \tag{5.1}$$

The initial and boundary conditions are taken to be

$$\begin{aligned} u(x, 0) &= -1 \qquad \text{for} \quad -0.5 \leqslant x \leqslant 0, \\ &= +1 \qquad \text{for} \quad 0 \leqslant x < 0.5. \end{aligned}$$

The exact solution is given to a good approximation by the steady state

$$u = \tanh(x/2\nu). \tag{5.2}$$

All the solutions discussed in the present paper are obtained with 128 equally spaced grid points, $\Delta x = 1/128 = 0.007825$. We calculate the solution of (5.1) with the difference scheme (4.2.6), which can be written in the form

$$\begin{aligned} &\left(\frac{\Delta t}{12\,\Delta x} u_m^* + \frac{\Delta t}{12\,\Delta x} u_{m-1}^* - \frac{\nu \Delta t}{2\,\Delta x^2}\right) u_m^{m+1} + \left(1 + \nu \frac{\Delta t}{\Delta x^2}\right) u_m^{n+1} \\ &\quad - \left(\frac{\Delta t}{12\,\Delta x} u_m^* + \frac{\Delta t}{12\,\Delta x} u_{m+1}^* + \frac{\nu \Delta t}{2\,\Delta x^2}\right) u_{m+1}^{n+1} = f_m^n, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} f_m^n = u_m^n &+ \frac{\Delta t}{6} \left[\left(u_m^* \frac{u_{m+1}^* - u_{m-1}^*}{2\,\Delta x} \right) + \frac{u_{m+1}^* u_{m+1}^n - u_{m-1}^* u_{m-1}^n}{2\,\Delta x} \right] \\ &+ \frac{\nu \Delta t}{2\,\Delta x^2} (u_{m+1}^n - 2u_m^n + u_{m-1}^n) \end{aligned}$$

TABLE I

	$\nu = 0.001$		$\nu = 0.002$		$\nu = 0.003$		$\nu = 0.004$		$\nu = 0.005$		$\nu = 0.01$	
	u	$\tanh(x/2)$	u	$\tanh(x/2)$	u	$\tanh(x/2)$	u	$\tanh(x/2)$	u	$\tanh(x/2)$	u	$\tanh(x/2)$
20	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000
19	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−0.999999	−0.999999
18	−0.999999	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−0.999998	−0.999998
17	−1.000001	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−0.999997	−0.999995
16	−0.999998	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−0.999994	−0.999989
15	−1.000004	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−0.999986	−0.999976
14	−0.999856	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−0.999967	−0.999947
13	−1.000240	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−0.999926	−0.999885
12	−0.999591	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−0.999830	−0.999749
11	−1.000691	−1.000000	−0.999998	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−0.999612	−0.999053
10	−0.998833	−1.000000	−1.000008	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−0.999999	−0.999115	−0.998805
9	−1.001968	−1.000000	−0.999977	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−1.000000	−0.999997	−0.997982	−0.997391
8	−0.996674	−1.000000	−1.000073	−1.000000	−1.000000	−1.000000	−1.000000	−0.999999	−0.999999	−0.999984	−0.995403	−0.994310
7	−1.005604	−1.000000	−0.999775	−1.000000	−0.999999	−1.000000	−1.000000	−0.999994	−0.999995	−0.999922	−0.989547	−0.987614
6	−0.990508	−1.000000	−1.000698	−1.000000	−1.000008	−0.999999	−1.000000	−0.999957	−0.999957	−0.999630	−0.976342	−0.973143
5	−1.015938	−1.000000	−0.997834	−1.000000	−0.999945	−0.999983	−0.999999	−0.999695	−0.999647	−0.998234	−0.946998	−0.942258
4	−0.972844	−1.000000	−1.006678	−0.999980	−1.000417	−0.999780	−0.999983	−0.997853	−0.997138	−0.991602	−0.883845	−0.878050
3	−1.045149	−0.999999	−0.979074	−0.999885	−0.996813	−0.997029	−0.998630	−0.984963	−0.977392	−0.960562	−0.756522	−0.751574
2	−0.921705	−0.999984	−1.062746	−0.994310	−1.023582	−0.960562	−0.938393	−0.898589	−0.844078	−0.824872	−0.528954	−0.526968
1	−1.126683	−0.960562	−0.779015	−0.751574	−0.578432	−0.572371	−0.454639	−0.452851	−0.372553	−0.371899	−0.192890	−0.192866

$$u_m^* = u_m^n + \frac{\Delta t\,\beta}{3}\left[\left(u_m^n \frac{u_{m+1}^n - u_{m-1}^n}{2\Delta x}\right) + \frac{(u_{m+1}^n)^2 - (u_{m-1}^n)^2}{2\Delta x}\right]$$
$$+ \frac{\nu\beta\,\Delta t}{\Delta x^2}(u_{m+1}^n - 2u_m^n + u_{m-1}^n).$$

In matrix form, we have $Au^{n+1} = u^n$, where $A = I + \Delta t\, R$, R is an $N \times N$ matrix depending on Δx and U^n, and I is the unit matrix. It is not difficult to show that $(AU, U) = \|U\|^2$. Thus by Schwarz's inequality, we have $\|AU\| \geqslant \|U\|$. Hence A is invertible and in practice we solve $Au^{n+1} = u^n$ with a band solver.

Computations were done on machine model 013 of the Computer Center of the Academia Sinica. Numerical solutions of system (5.3) have been obtained for ($\beta = \frac{1}{2}$, $\alpha = \frac{1}{2}$) $\nu = 0.001$, 0.002, 0.003, 0.004, 0.005, and 0.01. See Table I. We list only half of the table because the solution is antisymmetric. As our time step we select $\Delta t = 0.01$.

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